Application of BCS-like Ideas to Superfluid 3-He

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Electrons in Metals (BCS):
Fermions of spin $\frac{1}{2}$, $T_F \sim 10^4 K$, $T_c \sim 10 K$
⇒ strongly degenerate at onset of superconductivity
Normal state: in principle described by Landau Fermi-liquid theory, but “Fermi-liquid” effects often small and generally very difficult to see.

BCS: model normal state as
Weakly interacting gas with weak “fixed”
   attractive interaction

Liquid $^3$He:
also fermions of spin $\frac{1}{2}$, $T_F \sim 1K$, $T_c \sim 10^{-3} K$
⇒ again, strongly degenerate at onset of superfluidity
Normal state: must be described by Landau Fermi-liquid theory, Fermi-Liquids effects very strong. (e.g. Wilson ratio $\sim 4$)
⇒ low-lying states (inc. effects of pairing) must be described in terms of Landau quasiparticles.

What is Common:
2-particle density matrix has single macroscopic ($\sim N$) eigenvalue, with associated eigenfunction

$$ F(r_1 r_2 \sigma_1 \sigma_2) \equiv F(R : r \sigma_1 \sigma_2) $$

“wave function of Cooper pairs”
(for $r, \sigma_1, \sigma_2$ fixed: GL “macroscopic wave function $\Psi(R)$)
**STRUCTURE OF COOPER-PAIR WAVE FUNCTION**

(in original BCS theory of superconductivity, for fixed $\mathbf{R}, \sigma_1, \sigma_2$)

$$F(\mathbf{r}) = F(r) = \Delta \sum_k (2E_k)^{-1} \exp \left( i \mathbf{k} \cdot \mathbf{r} \right) \left( \epsilon_k^2 + |\Delta|^2 \right)^{1/2}$$

Energy gap

$$\cong \text{const.} \left( N\Delta / E_F \Omega^{1/2} \right) \frac{\sin \frac{k_FR}{r}}{\frac{k_FR}{r}} \exp \left( -\frac{r}{\xi} \right)$$

$$\xi = \text{"pair radius"} \sim \hbar \nu_F / \Delta \left( \sim 10^4 \text{Å} \right)$$

"Number of Cooper pairs" $(N_0) = \text{norm}^n$ of $F(r)$

$$\equiv \int |F(\mathbf{r})|^2 \, d\mathbf{r} \sim \frac{N^2}{\Omega} \frac{\Lambda^2}{E_F^2} \frac{1}{k_F^2} \xi \sim N \left( \frac{\Delta}{E_F} \right) \sim 10^{-4} N$$

(cf: $N_0 / N \sim 10\%$ in $^4\text{He}$)

In original BCS theory of superconductivity,

$$F(\mathbf{r} : \sigma_1 \sigma_2) = \frac{1}{\sqrt{2}} \left( \uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2 \right) F(|\mathbf{r}|)$$

spin singlet orbital s-wave

$\Rightarrow$PAIRS HAVE NO "ORIENTATIONAL" DEGREES OF FREEDOM

$(\Rightarrow$stability of supercurrents, etc.)
THE FIRST ANISOTROPIC COOPER-PAIRED SYSTEM: SUPERFLUID $^3$He

2-PARTICLE DENSITY MATRIX $\rho_2$ still has one and only one macroscopic eigenvalue
$\Rightarrow$ can still define “pair wave function” $F(R,r;\sigma_1\sigma_2)$

However, even when $F \neq F(R)$,

$F(r;\sigma_1\sigma_2)$ HAS ORIENTATIONAL DEGREES OF FREEDOM!
(i.e. depends nontrivially on $\hat{r},\sigma_1\sigma_2$)

All three superfluid phases have $\ell = S = 1$

A phase ("ABM")

Spin triplet

$F(r;\sigma_1\sigma_2) = \frac{1}{\sqrt{2}}(\uparrow \downarrow_x + \downarrow \uparrow_2) \times f(r)$

$\Rightarrow$ char. "spin axis"

Properties anisotropic in orbital and spin space separately,

e.g. $|\Delta_k| = |\Delta(\hat{k})| = \Delta_n |\hat{k} \times \hat{\ell}| \Leftarrow$ nodes at $\pm \hat{\ell}$!

WHAT IS TOTAL ANG. MOMENTUM? $\left\{ \frac{N}{N} \frac{\Delta / E_f}{\Delta / E_f^2} \right\}$?
**B phase ("BW")**

For any particular direction $\hat{n}$ (in real or $k$-space) can always choose spin axis s.t.

$$F(\hat{n} : \sigma_i \sigma_j) \sim \frac{1}{\sqrt{2}} \left( \uparrow_i \downarrow_j + \downarrow_i \uparrow_j \right) \hat{d}$$

i.e. $\hat{d} = \hat{d}(\hat{n})$. Alternative description:

BW phase is $^3P_0$ state “spin-orbit rotated” by $104^\circ$.

$L = S = J = 0$ because of dipole force $\cos^{-1}(-1/4) = \theta_0$

Note: rotation (around axis $\hat{\omega}$) breaks $P$ but not $T$

Orbital and spin behavior individually isotropic, but: properties involving spin-orbit correlations anisotropic!

Example: NMR

$$\frac{dS}{dt} = S \times H_O + \frac{\delta E_D}{\delta \theta}$$

$\angle$ of rotation about rf field direction $\hat{\mathcal{H}}_{rf}$

In transverse resonance, rotation around $\hat{\mathcal{H}}_{rf}$ equiv. rotation of $\hat{\omega}$ with $\theta_0$ unchanged

$\Rightarrow$ No dipole torque.

In longitudinal resonance, rotation changes $\theta_0$

$\Rightarrow$ finite-frequency resonance!
RESOLUTION OF THE PARADOX OF TWO NEW PHASES.


In BCS (weak-coupling) theory for $\ell = 1$, BW phase is always stable, independently of pressure and temperature.

Crucial difference between Cooper pairing in superconductors and $^3\text{He}$:

Superconductor:

![Diagram of superconductor pairing process]

liquid $^3\text{He}$:

![Diagram of $^3\text{He}$ pairing process]

$\Rightarrow$ “feedback” effects: Over most of the phase diagram, BW state stable as in BCS theory. But at high temperature and pressure, feedback effects uniquely favor ABM phase.
“Exotic” Properties of Superfluid $^3$He

A. Orientation const. in space, varying in time:
   — spin dynamics (NMR)
   — orbital dynamics (“normal locking”) (A phase)
   — effect of macroscopic ang. momentum? (A phase)

B. Orientation const. in time, varying in space
   — spin textures ($^3$He-A) ($\hat{d} = \pm \hat{l}$) in equation
     \[
     \hat{\mathbf{d}} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
     \hat{\mathbf{\ell}} \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
     \text{(carries spin current)}
     \]
   — orbital textures
   — topological singularities (boojums, “half-quantum” vortices.
   — instability of supercurrents in $^3$He-A.

C. Orientation varying in both space and time
   — spin waves
   — orbital waves
   — “flapping” and “clapping” modes

D. Amplification of ultra-weak effects
**SPONTANEOUSLY BROKEN SPIN-ORBIT SYMMETRY**

Ferromagnetic analogy:

**FERROMAGNET**

\[ \hat{H} = \hat{H}_0 + \hat{H}_z \]

\[ \uparrow \]

invariant under simult. rotation of all spins

\[ \hat{H}_z = - \mu_B \mathcal{H} \sum_i S_{zi} \]

breaks spin-rot. symmetry

Exlt. field

Paramagnetic phase (T > T_c):
spins behave independently,
kT competes with \( \mu_B \mathcal{H} \) \( \Rightarrow \)
polarization \( \sim \mu_B \mathcal{H} / kT \ll 1 \) \( \Rightarrow \)
\[ <H_z> \sim N(\mu_B \mathcal{H})^2 / kT \]

Ferromagnetic phase (T < T_c):
\( \hat{H}_0 \) forces all spins to lie parallel
\( \Rightarrow k_B T \) competes with \( N \mu_B \mathcal{H} \)
\( \Rightarrow <S_z> \sim 1 \) \( \Rightarrow <H_z> \sim N \mu_B \mathcal{H} \)

**LIQUID ^3 HE**

\[ \hat{H} = \hat{H}_0 + \hat{H}_D \]

\[ \uparrow \]

invariant under relative rotation of spin + orbital coordinate systems

\[ \equiv \mu_n \mathcal{H}_n / r_0 \]

\[ \hat{H}_D = g_D \sum_{ij} \left( \mathbf{\sigma}_i \cdot \mathbf{\sigma}_j - \frac{3}{2} \mathbf{\sigma}_i \cdot \mathbf{L}_{ij} \mathbf{\sigma}_j \cdot \mathbf{L}_{ij} \right) \]

\[ \left( \frac{3}{r_{ij}^3} \frac{3}{r_0^3} \right) \]

breaks relative spin-orbit rot. symmetry

Normal phase (T > T_A):
pairs of spins behave independently \( \Rightarrow \)
polarization \( \sim g_D / kT \ll 1 \) \( \Rightarrow \)
\[ <H_D> \sim N g_D ^2 / kT \]

Ordered phase (T < T_A):
\( \hat{H}_0 \) forces all pairs to behave similarly \( \Rightarrow \)
kT competes with \( N g_D \)
\[ \Rightarrow <H_D> \sim N g_D ^5 \sim 10^{-3} \text{ ergs/cm}^3 ! \]
SBSOS: ORDERING MAY BE SUBTLE

FERROMAGNET

\[ \downarrow \]

\[ \uparrow \uparrow \uparrow \uparrow \uparrow \]

\[ \downarrow \]

\[ \uparrow \uparrow \uparrow \uparrow \uparrow \]

\[ \leftarrow \text{NORMAL PHASE} \rightarrow \]

\[ \leftarrow \text{ORDERED PHASE} \rightarrow \]

LIQUID $^3$HE

\[ \uparrow \uparrow \uparrow \uparrow \]

\[ \uparrow \downarrow \downarrow \downarrow \]

\[ \downarrow \uparrow \uparrow \uparrow \uparrow \]

\[ \downarrow \uparrow \uparrow \uparrow \uparrow \]

\[ \leftarrow \text{NORMAL PHASE} \rightarrow \]

\[ \leftarrow \text{ORDERED PHASE} \rightarrow \]

\( \langle S \rangle \neq 0 \)

(\( \uparrow \) = total spin of pair
\( \downarrow \) = relative orbital ang. momentum)

\( \langle S \rangle = \langle L \rangle = 0 \)

but \( \langle L \times S \rangle \neq 0! \)
Amplification of ultra-weak effects (cf NMR):
Example: P- (but not T-) violating effects of neutral current part of weak interaction:
For single elementary particle, any EDM \( d \) must be of form
\[
\hat{d} = \text{const. } \mathcal{J} \leftarrow \text{violates T as well as P.}
\]
But for \(^3\text{He} - \text{B}\), can form
\[
d \sim \text{const. } \mathcal{L} \times \mathcal{S} \sim \text{const. } \hat{\Theta}
\]
violates P but not T.

Effect is tiny for single pair, but since all pairs have same value of \( \mathcal{L} \times \mathcal{S} \), is multiplied by factor of \( \sim 10^{23} \) ⇒

macroscopic P-violating effect?